CHAPTER 1

Introduction

1.1 Introduction and historical background

In his 1870 *Traité des substitutions et des équations algébriques*, one of the questions Camille Jordan considered was the classification of "the general types of equations solvable by radicals" [**47**, p. VIII]. He interpreted this as equivalent to the solution of the following problem [**47**, p. 396].

PROBLEM A. *Let $N > 0$ be an integer. Classify the transitive maximal solvable subgroups of the symmetric group S_N.*

(Here "maximal solvable" means maximal among the solvable subgroups of S_N, with respect to inclusion.) The main difficulty in Problem A lies in the classification of primitive maximal solvable subgroups, which Jordan reduced to the following two problems.

PROBLEM B. *Let $n > 0$ be an integer and let p be a prime. Classify the maximal irreducible solvable subgroups of* $\mathrm{GL}_n(p)$.

PROBLEM C. *Let $n > 0$ be an integer and let p be a prime. Classify the maximal irreducible solvable subgroups of* $\mathrm{GSp}_{2n}(p)$, $\mathrm{O}^+_{2n}(2)$, *and* $\mathrm{O}^-_{2n}(2)$.

Jordan was able to solve these problems, which he later wrote was "l'objet principal" of his *Traité* [**52**, p. 263]. He achieves this in the last part of the *Traité*, Livre IV, which takes up nearly 300 pages of the book. The solution given by Jordan is what Dieudonné called a "gigantesque recurrencé" that proceeds by a series of successive reductions [**16**, p. XXXIV]. Essentially, Jordan identifies the various types of maximal solvable subgroups, and describes them in terms of groups of smaller degree. For example, imprimitive groups are described in terms of primitive groups in smaller degree.

After Jordan, these types of reductions have become a very basic and useful technique — recall for example the O'Nan-Scott theorem on maximal subgroups of symmetric groups [**60**], [**4**], [**70**]. One can also see the solution of Problems B and C as a predecessor to the classification theorems of Aschbacher, Kleidman–Liebeck, and Bray–Holt–Roney-Dougal on maximal subgroups of classical groups [**2**], [**55**], [**7**].

The results of Livre IV in Jordan's *Traité* seem to have received less attention than some of his other results. Jordan himself wrote later that the proof is "confusing" and contains some mistakes [**52**, p. 264]. Much later in his life (1908 and 1917), Jordan published two papers where he gives a more clear presentation of his results [**51, 52**].

One purpose of this book is expository: we give a proof of Jordan's classification in modern terms, for a large part based on the ideas in [**51, 52**]. More generally, we will use Jordan's methods to classify the maximal irreducible solvable subgroups of $\mathrm{GL}_n(\mathbb{F})$, $\mathrm{GSp}_{2n}(\mathbb{F})$, and $\mathrm{GO}_n^{\varepsilon}(\mathbb{F})$ over a finite field $\mathbb{F}$. Previous work in this direction was done by D. A. Suprunenko in the 1950s [**76**], who studied solvable and nilpotent subgroups of $\mathrm{GL}_n(\mathbb{F})$ over an arbitrary field $\mathbb{F}$. Some of the ideas used by Suprunenko are similar to those of Jordan, but take advantage of the basic methods and language of representation theory.

We will now give an outline of Jordan's classification of maximal solvable subgroups, see also the survey by Dieudonné [**16**]. In [**51, 52**] Jordan provides a solution to the following slightly more general problem.

PROBLEM A'. *Let $N > 0$ be an integer. Classify the maximal solvable subgroups of the symmetric group S_N.*

For the classification of maximal solvable subgroups $G \leq S_N$, the first step is to reduce the problem to the case where G is transitive. If G is intransitive, it is easily seen that $G = G_1 \times \cdots \times G_t$, where G_i is maximal transitive solvable in S_{n_i}, and $n_1 + \cdots + n_t = N$. Jordan determines when such a group $G = G_1 \times \cdots \times G_t$ is maximal solvable in S_N, thus reducing the problem to the case where G is transitive.

Similarly, if G is transitive and imprimitive, then G is equal to a wreath product $G_1 \wr G_2 \wr \cdots \wr G_t$, where G_i is transitive primitive maximal solvable in S_{n_i}, and $N = n_1 n_2 \cdots n_t$ with each $n_i > 1$ equal to some prime power. Jordan proves that such a wreath product is maximal solvable, except when $(n_i, n_{i+1}) = (2, 2)$ for some i, which has to be excluded since $S_2 \wr S_2 \lneqq S_4$. The problem is thus reduced to the primitive case.

When G is primitive maximal solvable, we have $N = p^n$ for some prime p and G is an affine group of the form $G = \mathbb{F}_p^n \rtimes X$, where $X \leq \mathrm{GL}_n(p)$ is maximal irreducible solvable. It turns out such a group is always maximal solvable in S_N, so the following theorems hold.

THEOREM 1.1.1 (Jordan). *Let G be a maximal solvable subgroup of S_N. Then G is of one of the following types:*

Type (I): *G is intransitive, and the following hold:*
- (a) *$N = n_1 + n_2 + \cdots + n_t$, where $t \geq 2$ and $n_i \geq 1$ for all $1 \leq i \leq t$;*
- (b) *$G = G_1 \times G_2 \times \cdots \times G_t$, where G_i is a maximal transitive solvable subgroup of S_{n_i} for all $1 \leq i \leq t$;*
- (c) *$G_i \ncong G_j$ as permutation groups for all $i \neq j$;*
- (d) *$G_i \ncong G_j \wr S_2$ as permutation groups for all $i \neq j$;*
- (e) *$G_i \ncong G_j \wr S_3$ as permutation groups for all $i \neq j$;*

Type (II): *G is transitive and imprimitive, and the following hold:*
- (a) *$N = n_1 n_2 \cdots n_t$, where $t \geq 2$ and $n_i > 1$ is a prime power for all $1 \leq i \leq t$;*
- (b) *$G = G_1 \wr G_2 \wr \cdots \wr G_t$, where G_i is a maximal primitive solvable subgroup of S_{n_i} for all $1 \leq i \leq t$;*
- (c) *$(n_i, n_{i+1}) \neq (2, 2)$ for all $1 \leq i < t$.*

Type (III): *G is primitive, and the following hold:*
- (a) *$N = p^n$ for some prime p and integer $n > 0$;*
- (b) *G is an affine group $G = \mathbb{F}_p^n \rtimes X$, where $X \leq \mathrm{GL}_n(p)$ is maximal irreducible solvable.*

THEOREM 1.1.2 (Jordan). *Let $G \leq S_N$ be one of the Types (I) – (III) in Theorem 1.1.1. Then G is maximal solvable in S_N.*

We will give a proof of Theorems 1.1.1 and 1.1.2 in Section 1.3, following Jordan [**52**]. Here the proof of Theorem 1.1.1 is straightforward. For Theorem 1.1.2, the key observation is that in a solvable primitive permutation group, a nontrivial element fixes at most half of the points (Lemma 1.3.5).

Problem A' is then reduced to the classification of maximal irreducible solvable subgroups of $\mathrm{GL}_n(p)$ (Problem B), which is where the main difficulties arise.

First one needs to narrow down the structure of maximal irreducible solvable subgroups. This is attained by Jordan in [**51**], where he describes and constructs the possible candidates for the maximal irreducible solvable subgroups G of $\mathrm{GL}_n(p)$. As a first step, if such a G is imprimitive, there exist integers $k > 1$ and $d > 0$ such that $n = dk$ and $G = G_0 \wr X$, where $G_0 \leq \mathrm{GL}_d(p)$ is primitive maximal irreducible solvable, and $X \leq S_k$ is maximal transitive solvable.

Thus for the construction of maximal irreducible solvable subgroups, it is not difficult to reduce to the case where G is primitive. In this case Jordan first considers a maximal abelian normal subgroup $F \trianglelefteq G$, which he calls the *premier faisceau* of G.

In modern terms, it turns out that the $\mathbb{F}_p$-algebra generated by F is a finite field $\mathbb{K} \cong \mathbb{F}_{p^\nu}$, where $n = \mu\nu$ for some integer $\mu \geq 1$. Furthermore, we have $F = \mathbb{K}^\times$, so F is cyclic of order $p^\nu - 1$. If $\mu = 1$, then F is generated by a Singer cycle, and G is equal to the Singer cycle normalizer $\Gamma\mathrm{L}_1(p^n)$ of order $n(p^n - 1)$.

Suppose then that $\mu > 1$. In this case, Jordan shows that G can be constructed in terms of maximal irreducible solvable subgroups of general symplectic groups $\mathrm{GSp}_{2\ell}(r)$ and $\mathrm{O}^+_{2\ell}(2)$. Roughly speaking, Jordan shows that we have a factorization

$$\mu = r_1^{\ell_1} \cdots r_k^{\ell_k},$$

where r_i are primes (with possibly some of the r_i being equal), such that $r_i \mid p^\nu - 1$ for all $1 \leq i \leq k$, and such that G can be constructed in terms of $X_1, \ldots, X_k$, where:

- X_i is a maximal irreducible solvable subgroup of $\mathrm{GSp}_{2\ell_i}(r_i)$ if $r_i > 2$;
- X_i is a maximal irreducible solvable subgroup of $\mathrm{O}^+_{2\ell_i}(2)$ or $\mathrm{O}^-_{2\ell_i}(2)$ if $r_i = 2$.

To be a bit more specific, one can show that $A = \mathrm{Fit}(C_G(F))$ is an absolutely irreducible subgroup of $C_{\mathrm{GL}_n(q)}(F) = \mathrm{GL}_\mu(\mathbb{K})$, and decomposes as a tensor product $A = R_1 \otimes \cdots \otimes R_k$, where $R_i \trianglelefteq G$ is an extraspecial r_i-group of order $r_i^{1+2\ell_i}$ and exponent $r_i \gcd(r_i, 2)$. This provides a homomorphism

$$\pi : N_{\mathrm{GL}_n(p)}(F, R_1, \ldots, R_k) \to \prod_{i=1}^{k} \mathrm{GSp}_{2\ell_i}(r_i)$$

defined by $\pi(g) = (g_1, \ldots, g_k)$, where g_i is the action of g on $R_i/Z(R_i)$. It turns out that $G = \pi^{-1}(X_1 \times \cdots \times X_k)$.

Conversely, for any such factorization $\mu = r_1^{\ell_1} \cdots r_k^{\ell_k}$ and groups $X_1, \ldots, X_k$, one can construct $\pi^{-1}(X_1 \times \cdots \times X_k)$, which turns out to be an irreducible primitive solvable subgroup of $\mathrm{GL}_n(p)$. Thus the main question is reduced to the classification of maximal irreducible solvable subgroups of $\mathrm{GSp}_{2\ell}(r)$ and $\mathrm{O}^\pm_{2\ell}(2)$, in other words, to Problem C.

Using similar reductions — starting again from the imprimitive case — Jordan shows that Problem C can also be reduced to groups of smaller degree. With this the construction of all maximal solvable subgroups of S_N is complete. At the bottom of this massive recursion, we have the normalizers $\Gamma L_1(p^n)$ of Singer cycles in $\mathrm{GL}_n(p)$, and their analogues in $\mathrm{GSp}_{2\ell}(r)$ and $\mathrm{O}^{\pm}_{2\ell}(2)$.

In Section 2.1 – Section 4.3, we will similarly analyze the structure of maximal irreducible solvable subgroups of $\mathrm{GL}_n(q)$, $\mathrm{GSp}_{2\ell}(q)$ and $\mathrm{GO}^{\varepsilon}_n(q)$, for every prime power q. Using the basic approach of Jordan with some adjustments, we will find similarly to Jordan that all such groups are constructed recursively in terms of maximal irreducible solvable subgroups of smaller degree. The base of the recursion again consists of normalizers of Singer cycles in $\mathrm{GL}_n(q)$, and their analogues in symplectic and orthogonal groups.

The recursive construction provides an algorithm to construct a list of groups which contains all maximal irreducible solvable subgroups. It then remains to check which of the subgroups constructed are actually maximal solvable. This question is the topic of [**52**], where Jordan gives a proof that apart from a few families of examples, the construction always provides maximal solvable subgroups. In Chapter 7, we will prove a similar classification result for $\mathrm{GL}_n(q)$, $\mathrm{GSp}_{2\ell}(q)$, and $\mathrm{GO}^{\pm}_n(q)$ for every prime power q. As a necessary part of this classification, we will also classify maximal irreducible solvable subgroups of $\mathrm{Sp}_{2\ell}(q)$ and $O^{\pm}_n(q)$ for every prime power q. Furthermore, for q even, we will classify maximal irreducible solvable subgroups of $\Omega^{\pm}_{2\ell}(q)$.

The first step in the classification is to verify some basic properties of the groups given by the construction. For example, one needs to prove that the subgroups $\pi^{-1}(X_1 \times \cdots \times X_k)$ mentioned earlier are indeed irreducible and primitive. We establish results of this type in Chapter 5. After this, two key results that are needed in the classification are the following.

THEOREM 1.1.3. *Let $G \leq \mathrm{GL}_n(q)$ be primitive irreducible solvable. Then for every $g \in G \setminus \{1\}$, the fixed point space of g on $\mathbb{F}_q^n$ has dimension $\leq 3n/4$.*

THEOREM 1.1.4. *Let $G \leq \mathrm{GL}_n(q)$ be irreducible and solvable. If D is an abelian subgroup of the affine group $\mathbb{F}_q^n \rtimes G$, then $|D| \leq q^n$.*

Jordan proved these results in the case where $q = p$ is a prime, but as we shall see in Chapter 6, the basic idea of his proof works for every prime power q.

In fact, for Theorem 1.1.3 Jordan provides a more precise upper bound for elements of prime order, which we will also need. In Jordan's proof, the main argument consists essentially of finding such an upper bound for elements in normalizers of extraspecial groups in $\mathrm{GL}_n(q)$ (Aschbacher class $\mathcal{C}_6$). We will prove the following result in Section 6.1.

THEOREM 1.1.5. *Suppose that $R \leq \mathrm{GL}_n(q)$ is absolutely irreducible, where R is an extraspecial r-group of exponent r or 4. Let Z be the group of scalar matrices in $\mathrm{GL}_n(q)$.*

Let $g \in N_{\mathrm{GL}_n(q)}(RZ)$ be an element of prime order ϖ and suppose that g is non-scalar. Let W be a g-eigenspace on $\mathbb{F}_q^n$. Then

$$\dim W \leq \begin{cases} 3n/4, & \text{if } \varpi = 2, \\ 2n/3, & \text{if } \varpi = 3, \\ n/2, & \text{if } \varpi > 3. \end{cases}$$

Moreover, if $\varpi = 3$ and n is not a multiple of 3, then we also have $\dim W \leq n/2$.

Some similar bounds were given by Guralnick and Maróti in [**31**, Section 2], with a different proof that generalizes a result of Hall and Higman [**33**, Theorem 2.5.1]. Furthermore, Theorem 1.1.3 and the corresponding result for normalizers of extraspecial groups have appeared and have been applied many times in the literature; see for example [**26**, proof of Proposition 4], [**72**, Lemma 2.3], [**32**, p. 452], [**10**, Lemma 6.3].

Using the results established in Section 4.4 – Section 6.3, we complete the classification of maximal irreducible solvable subgroups in Chapter 7. More generally in the case of $\mathrm{GSp}_{2\ell}(q)$ and $\mathrm{GO}_n^{\varepsilon}(q)$, in Section 7.5 we will also classify *metrically completely reducible* maximal solvable subgroups, where metrically completely reducible means that the group has no nonzero invariant subspaces which are totally isotropic. We will also classify metrically completely reducible maximal solvable subgroups in $\mathrm{Sp}_{2\ell}(q)$, $O_n^{\varepsilon}(q)$, and if q is even, in $\Omega_{2\ell}^{\pm}(q)$.

Finally at the end of this book in Chapter 8, we illustrate our results by providing tables of maximal solvable subgroups in small degrees.

Remark 1.1.6. From the recursive construction presented in this book, one can extract an efficient algorithm for finding generators for the maximal solvable subgroups of S_N, and for the maximal irreducible solvable subgroups of the classical groups that we consider. This could be implemented in a computer algebra system such as GAP or Magma, however we have not written down the algorithm precisely in this book.

Previously an algorithm for constructing maximal irreducible solvable subgroups of $\mathrm{GL}_n(q)$ was proposed in [**23**, Section 4]. The algorithm in [**23**] first constructs a list of candidates for maximal irreducible solvable subgroups based on Aschbacher's theorem, and then checks which of these subgroups are maximal solvable in $\mathrm{GL}_n(q)$. However, this approach has issues in certain degrees — see Remark 5.5.16.

Remark 1.1.7 (Historical background). We finish this introduction by providing some more historical background to Jordan's classification of maximal solvable subgroups of symmetric groups.

The origin of group theory is in the study of algebraic equations. In the early 1820s, Niels Henrik Abel (1802–1829) proved that in general, the quintic equation is not solvable by radicals [**54**, p. 67]. Later in 1826, Abel stated in a letter that he was working on "determining the form of all the algebraic equations which can be solved algebraically", and he believed that he would be able to solve this more general problem [**1**, p. 256, p. 260] [**54**, p. 71] [**83**, p. 101].

Abel died in 1829 and did not have chance to finish his work in this direction, but he did obtain results in some special cases. In modern terms, one of his results is that a polynomial equation $f(X) = 0$ is solvable by radicals if the corresponding Galois group is commutative [**54**, p. 71].

The general result was given by Évariste Galois (1811–1832), who proved (again in modern terms) that a polynomial equation $f(X) = 0$ over a field of characteristic zero is solvable by radicals if and only if the corresponding Galois group is solvable. This result appeared in the *Premiere mémoire* that Galois had submitted to the Académie des Sciences, but which was rejected [**22**]. Galois' paper was unclear, and the referee report by Poisson and Lacroix from 1831 suggested that Galois should develop his work further:

> "His reasoning is neither clear enough nor well enough developed for us to have been able to judge its correctness [...] The author announces that the proposition which forms the special goal of his memoir is part of a general theory susceptible of many other applications. [...] One may therefore wait until the author will have published his work in its entirely before forming a final opinion; but given the present state of the part that he has submitted to the Academy, we cannot propose to you that you give it your approval." (Translation from [**68**, IV.2, pp. 148–149])

Galois died in a duel in 1832, and it took a long time before his ideas were fully developed. The *Premiere mémoire* and another unpublished manuscript of Galois were published posthumously in 1846 by Liouville [**63**]. From here the ideas of Galois spread and were appreciated by famous mathematicians such as Betti, Dedekind, Jordan, Kronecker, and Serret [**83**, pp. 118 – 135], [**68**, I.5]. For example, Betti [**6**] and Serret [**73**] wrote texts which contained expositions of Galois' results, correcting mistakes and filling in missing details from the terse and incomplete works of Galois.

The first significant developments on Galois' work were made by Jordan. Starting from his thesis [**41**], Jordan spent most of 1860–1870 working on topics related to algebraic equations, permutation groups, and solvable groups. This culminated in the publication of the *Traité* in 1870, which greatly expanded on the works of Galois and has had an enormous influence on group theory. As pointed out in [**67**, p. 414, 418], Jordan is overly modest in calling all the 667 pages of the *Traité* "just a commentary" on Galois' work [**47**, p. viii].

In the *Traité*, Jordan was particularly interested in Abel's problem of determining the different types of polynomial equations solvable by radicals, over a field F of characteristic zero [**47**, p. V, p. VIII]. As a starting point, Jordan takes the result of Galois that $f(X) \in F[X]$ is solvable by radicals if and only if the corresponding Galois group is solvable. To make this criterion more explicit, Jordan was led to consider the construction of maximal transitive solvable subgroups of S_N [**43**, p. 108] [**47**, p. 396].

For determining solvability by radicals, what Jordan essentially proposes in [**47**, p. 396] is the following approach. (This is a similar to how Jordan suggests [**47**, p. 276] that Galois groups can be computed, see [**13**, Section 13.3].)

It suffices to consider the irreducible case, so suppose that $f(X) \in F[X]$ is an irreducible polynomial of degree N, with N distinct roots $\alpha_1, \ldots, \alpha_N$ in some extension field of F. Let $G_1, \ldots, G_t$ be representatives for the conjugacy classes of maximal transitive solvable subgroups of S_N. The group corresponding to f is a transitive permutation group $G \leq S_N$ acting on the roots of f. Thus by Galois' result, we know that f is solvable by radicals if and only if G is contained in a conjugate of some G_i.

To decide whether G is contained in a conjugate of G_i, start by finding a polynomial $\psi \in F[X_1, \ldots, X_N]$ such that G_i is the stabilizer of ψ in S_N. We consider the resolvent

$$R_\psi(Y, X_1, \ldots, X_N) := \prod_{j=1}^{M} (Y - \psi_j) \in F[Y, X_1, \ldots, X_N],$$

where $\psi_1, \ldots, \psi_M$ is the S_N-orbit of ψ. Then R_ψ is invariant under the action of S_N, so the coefficients of Y can be expressed in terms of elementary symmetric polynomials in $X_1, \ldots, X_N$. Therefore one can compute the specialized resolvent

$$R_\psi^f(Y) := R_\psi(Y, \alpha_1, \ldots, \alpha_N) \in F[Y]$$

using the coefficients of f.

At this point Jordan states [**47**, p. 276] (also in [**43**, p. 107], [**50**, footnote, p. 35]) that it suffices to check whether $R_\psi^f(Y)$ has a "rational root", meaning a root in F. As pointed out in [**13**, p. 386], this part of Jordan's argument is missing a detail, due to the possibility of rational roots which are not simple. What is true is the following (see for example [**13**, Proposition 13.3.2], or [**75**, Theorem 5]):

- If G is contained in a conjugate of G_i, then $R_\psi^f(Y)$ has a rational root.
- If $R_\psi^f(Y)$ has a simple rational root, then G is contained in a conjugate of G_i.

Thus if $R_\psi^f(Y)$ has rational roots and all of them occur with multiplicity greater than one, the method is inconclusive. However, we can fix this by modifying f by a suitable Tschirnhaus transformation to get another polynomial f_0, such that the Galois groups of f and f_0 are isomorphic as permutation groups, and $R_\psi^{f_0}(Y)$ is squarefree [**25**, Theorem 3, (2)].

Jordan comments that he sees his method describing solvability by radicals "satisfactory from a theoretical point of view", but that in general it leads to computations which are "impractical" [**43**, pp. 107 – 108] [**47**, pp. 276 – 277]. Indeed, already for small N the degrees of the polynomials involved become too large for calculations by hand. Currently one could implement Jordan's method in a computer algebra system, see for example [**75**] [**37**] [**24**] for computational aspects related to Galois groups and resolvents.

Thus Jordan interprets the question of solvability by radicals as equivalent to the classification of maximal transitive solvable subgroups S_N, stating that each maximal transitive solvable subgroup of S_N characterizes a type of equation solvable by radicals [**47**, p. 396]. Jordan was able to construct and classify maximal solvable subgroups of S_N, and later summarized that from his work "the problem of Abel is completely resolved" [**50**, p. 37].

As mentioned earlier in this introduction (Theorems 1.1.1 and 1.1.2), the classification of maximal transitive solvable subgroups of S_N can be reduced to the case of primitive groups. It was already known to Galois that a solvable primitive permutation group is of prime power degree p^n, and can be realized as a group of affine linear transformations over the finite field $\mathbb{F}_p$ of integers modulo p [**67**]. In modern terms, a solvable primitive permutation group of degree p^n (p prime) is an affine group $(\mathbb{F}_p)^n \rtimes X$, where $X \leq \mathrm{GL}_n(p)$ is irreducible and solvable.

In [**63**, p. 406], Galois makes a false claim which seems to be equivalent to the following statement: except for $p^n = 3^2$ and $p^n = 5^2$, every irreducible solvable subgroup of $\mathrm{GL}_n(p)$ is conjugate to a subgroup of the semilinear group $\Gamma\mathrm{L}_1(p^n)$. This result would seemingly solve the problem of describing maximal solvable subgroups, but turns out to be completely false, and Jordan points out Galois' mistake in many papers [**44**, p. 270], [**43**, p. 108], [**46**, p. 113], [**48**, p. 286].

The correct classification of maximal solvable subgroups is considerably more complicated, and takes up the entirety of Livre IV in the *Traité*. Jordan's solution proceeds by a massive recursion, which we have sketched earlier in this introduction. The original proof is somewhat difficult to follow and contains some mistakes, as Jordan himself writes in [**52**, p. 264].

Jordan also does not provide any concrete examples illustrating his construction in the *Traité*, although in [**46**] he demonstrates the classification for $\mathrm{GL}_2(p)$. In [**48**, Table A, p. 288], Jordan gives a table listing the number of maximal irreducible solvable subgroups of $\mathrm{GL}_n(p)$ for $p^n < 10^6$, but the table contains several mistakes and Jordan does not give a list of the groups themselves. For example, as pointed out in [**74**, p. 94], Jordan claims there are 5 classes of maximal irreducible solvable subgroups in $\mathrm{GL}_4(3)$, while the correct number is 4.

Despite its flaws, in Jordan's proof the key steps and techniques work, and lead to a solution of the problem. From the proof one can also extract intermediate results which are of independent interest, such as Theorem 1.1.4. Here is what Jordan states about mistakes in the *Traité*, in response to some criticism by Netto:

> "Nous ne saurions d'ailleurs avoir la prétention de n'avoir laissé se glisser aucune inexactitude dans un ouvrage aussi étendu que le nôtre, et qui traite un sujet nouveau et difficile; mais nous sommes persuadé qu'elles y sont en petit nombre."[1] ([**49**, p. 258], as quoted in [**66**])

Perhaps due to its difficulty at the time and a lack of clear exposition, Jordan's classification of maximal solvable subgroups received little attention during his lifetime. One attempt at deciphering Jordan's work is given by Bucht, who in a 96-page paper [**9**] goes through Jordan's classification in the cases of $\mathrm{GL}_3(p)$ and $\mathrm{GL}_4(p)$. Jordan published papers in 1908 and 1917 [**51, 52**] which give a more clearly organized and simplified version of the proof. A few gaps, errors, and unconsidered cases still remain in these papers — see for example Remarks 6.3.5, 4.1.8, 7.1.16, and 7.5.6.

Later, maximal solvable subgroups of linear groups have been studied by many authors, most notably Suprunenko in the 1950s and 1960s [**76, 77**]. Suprunenko studied (among other things) solvable and nilpotent subgroups of $\mathrm{GL}_n(\mathbb{F})$, where $\mathbb{F}$ is an arbitrary field or a division ring. He was certainly familiar with Jordan's results as he mentions in the introduction to [**76**], which also includes [**52**] in the bibliography.

In [**77**, §18 – §20], Suprunenko gives a description of the general structure of maximal solvable (not necessarily irreducible) subgroups of $\mathrm{GL}_n(\mathbb{F})$ over an arbitary field, generalizing results of Jordan. For the most part, Suprunenko does not attempt to study when the subgroups given by the construction are maximal solvable, although he does illustrate the results by giving a complete classification of maximal irreducible solvable subgroups of $\mathrm{GL}_r(q)$ for r prime [**77**, 21.3]. In [**78**] Suprunenko describes some other special cases, such as the maximal irreducible solvable subgroups of $\mathrm{GL}_4(p)$ for p prime.

Some other work discussing maximal solvable subgroups can be found in [**17**], [**74**], [**87**], and [**15**]. For more on the history surrounding Abel, Galois, and Jordan, see for example [**67**], [**68**], [**54**], [**83**], [**80**], [**8**].

[1] "We cannot, incidentally, claim that we have not let any inaccuracies slip into a work as extensive as ours, and which deals with a new and difficult subject; but we are convinced that they are there in small numbers."

1.2 Basic notation and terminology

Let r be a prime and q a power of a prime. Suppose that G, A, and B are finite groups. We use the following notation:

$\mathbb{F}_q$	Finite field with q elements
$\mathbb{F}^\times$	Multiplicative group of a field $\mathbb{F}$
$O_r(G)$	Largest normal r-subgroup of G
$O_{r'}(G)$	Largest normal r'-subgroup of G
$A.B$	Extension of A by B (normal subgroup A with quotient B)
$A \rtimes B$	Semidirect product of A by B (normal subgroup A with complement B)
$A \circ B$	Central product of A and B
$\delta_{i,j}$	Kronecker delta
S_N	Symmetric group of degree N
$\mathrm{Sym}(\Omega)$	Symmetric group on the set Ω
C_n	Cyclic group of order n
$\mathrm{Mat}_n(q)$	Set of $n \times n$ matrices with entries in $\mathbb{F}_q$
$\mathrm{GL}_n(q)$	General linear group of degree n over $\mathbb{F}_q$
X^g	Fixed point set $\{x \in X : gx = x\}$, for $g \in G$ with G acting on X
X^G	Fixed point set $\{x \in X : gx = x \text{ for all } g \in G\}$, for G acting on X

With notation such as $A.B$ and $A \rtimes B$ we do not specify the extension, so $G = A.B$ just means that G has a normal subgroup $N \cong A$ with $G/N \cong B$.

Let $a \in \mathbb{Z}$. The Legendre symbol is defined by

$$\left(\frac{a}{r}\right) = \begin{cases} 0, & \text{if } a \equiv 0 \mod r. \\ +1, & \text{if } a \text{ is a square modulo } r \text{ and } a \not\equiv 0 \mod r. \\ -1, & \text{if } a \text{ is not a square modulo } r. \end{cases}$$

We denote by ν_r the r-adic valuation on the integers, so if $a \neq 0$, then $\nu_r(a)$ is the largest integer k such that r^k divides a.

By a *form* on a $\mathbb{F}_q$-vector space we will mean a bilinear form, sesquilinear form, or a quadratic form.

If V is a $\mathbb{F}_q$-vector space equipped with a bilinear form b, for $W \subseteq V$ we denote by $W^\perp$ the subspace orthogonal to W. A subspace $W \subseteq V$ is *totally isotropic* if $W \subseteq W^\perp$, and *non-degenerate* if $W \cap W^\perp = 0$.

If b is an alternating bilinear form, we will also call totally isotropic subspaces *totally singular*. If V is equipped with a quadratic form Q, a totally isotropic subspace $W \subseteq V$ is *totally singular* if $Q(W) = 0$.

If W and W' are $\mathbb{F}_q$-vector spaces equipped with bilinear or sesquilinear forms b and b' respectively, then a *similarity* is a bijective linear map $g : W \to W'$ such that for some scalar λ, we have $b'(gv, gw) = \lambda b(v, w)$ for all $v, w \in W$. If $\lambda = 1$, then g is an *isometry*.

Similarly if W and W' are $\mathbb{F}_q$-vector spaces equipped with quadratic forms Q and Q' respectively, a *similarity* is a bijective linear map $g : W \to W'$ such that for some scalar λ, we have $Q'(gv) = \lambda Q(v)$ for all $v \in W$. If $\lambda = 1$, then g is an *isometry.*

Let κ be a form on a $\mathbb{F}_q$-vector space V. Then we denote:

$$\Delta(V,\kappa) = \{g \in \mathrm{GL}(V) : g \text{ is a similarity for } \kappa\}$$
$$I(V,\kappa) = \{g \in \mathrm{GL}(V) : g \text{ is an isometry for } \kappa\}$$

Let V and V' be $\mathbb{F}_q$-vector spaces equipped with forms κ and κ', respectively. Assume that either both κ and κ' are bilinear, or that they are both quadratic forms. We say that (V,κ) and (V',κ') are *similar* if there exists a similarity $V \to V'$. If there exists an isometry $V \to V'$, we say that (V,κ) and (V',κ') are *isometric.*

Let $H \leq \Delta(V,\kappa)$ and $K \leq \Delta(V',\kappa')$. Then H and K are said to be *similar* if there exists a similarity $g : V \to V'$ such that $gHg^{-1} = K$. We say that H and K are *isometric* if there exists an isometry $g : V \to V'$ such that $gHg^{-1} = K$.

Let V be a finite-dimensional vector space over $\mathbb{F}_q$ with $n = \dim V$. When n is even, there are two types of quadratic forms on V up to isometry, and the two types are distinguished by the dimension of a maximal totally singular subspace. For n odd there is a unique quadratic form on V up to a similarity. (See for example [**55**, Proposition 2.5.4].)

The type of a quadratic form Q on V is determined by the *signature*, which we define for n even as

$$\mathrm{sgn}(Q) = \begin{cases} +, & \text{if a maximal totally singular subspace has dimension } n/2. \\ -, & \text{if a maximal totally singular subspace has dimension } n/2-1. \end{cases}$$

For n odd, we define $\mathrm{sgn}(Q) = \circ$. Usually we will denote $\varepsilon = \mathrm{sgn}(Q)$, where $\varepsilon \in \{\circ, +, -\}$.

A bilinear form b is *reflexive* if $b(v,w) = 0$ implies $b(w,v) = 0$. It is well-known that a reflexive bilinear form must always be symmetric or alternating. Then if q is odd, for a reflexive bilinear form b we define $\mathrm{sgn}(b) = +$ if b is symmetric, and $\mathrm{sgn}(b) = -$ if b is alternating.

If b is a non-degenerate alternating bilinear form on V, we will denote

$$\Delta(V,b) = \mathrm{GSp}(V,b) = \mathrm{GSp}_n(q),$$
$$I(V,b) = \mathrm{Sp}(V,b) = \mathrm{Sp}_n(q).$$

If Q is a non-degenerate quadratic form on V with $\mathrm{sgn}(Q) = \varepsilon$, we denote

$$\Delta(V,Q) = \mathrm{GO}(V,Q) = \mathrm{GO}_n^\varepsilon(q),$$
$$I(V,Q) = O(V,Q) = O_n^\varepsilon(q).$$

In the case where n is odd and Q is a non-degenerate quadratic form on V, we will usually denote $\Delta(V,Q) = \mathrm{GO}_n(q)$ and $I(V,Q) = O_n(q)$.

If b is a non-degenerate Hermitian form on V, we denote

$$\Delta(V,b) = \Delta U(V,b),$$
$$I(V,b) = \mathrm{GU}(V,b).$$

1.3 Reduction to linear groups

In this section we will prove Theorems 1.1.1 and 1.1.2, which reduce the classification of maximal solvable subgroups of S_N to the classification of maximal irreducible solvable subgroups of $\mathrm{GL}_n(p)$, where p is a prime and $n > 0$ is an integer.

For the most part, the proofs of these two results only need some basic facts from permutation group theory, as found in standard textbooks such as [**81**] and [**18**]. We will use the following terminology.

DEFINITION 1.3.1. For a transitive permutation group $G \leq \mathrm{Sym}(\Omega)$, a *system of imprimitivity* is a collection $\{B_1, \ldots, B_k\}$ of subsets of Ω with $k > 1$, such that Ω is a disjoint union

$$\Omega = B_1 \cup \cdots \cup B_k$$

and G acts on $\{B_1, \ldots, B_k\}$. A system of imprimitivity is *trivial* if $|B_i| = 1$ for all $1 \leq i \leq k$. If G has a nontrivial system of imprimitivity, we say that G is *imprimitive*. Otherwise we say that G is *primitive*.

DEFINITION 1.3.2. Suppose that $G \leq \mathrm{Sym}(\Omega)$ is imprimitive, with systems of imprimitivity $\Omega = B_1 \cup \cdots \cup B_k$ and $\Omega = C_1 \cup \cdots \cup C_\ell$. We say that $\{B_1, \ldots, B_k\}$ is a *refinement* of $\{C_1, \ldots, C_\ell\}$ if each C_i is a union of some B_j's. If $\{B_1, \ldots, B_k\}$ has no proper nontrivial refinement, we say that $\{B_1, \ldots, B_k\}$ is *nonrefinable*.

The following lemma is probably well known, and is essentially a part of Jordan's proof of Theorem 1.1.2 in [**52**], see [**52**, §10].

LEMMA 1.3.3. *Suppose that $G \leq \mathrm{Sym}(\Omega)$ is imprimitive of the form $G = H \wr K$, where $H \leq S_d$ is primitive, $K \leq S_k$ is transitive, and $d, k > 1$. Then G has a unique nonrefinable system of imprimitivity.*

PROOF. Let $\Omega = B_1 \cup \cdots \cup B_k$ be the system of imprimitivity defining G. We have $G = (H_1 \times \cdots \times H_k) \rtimes K$, where H_i acts trivially on B_j for $j \neq i$, and the action of H_i on B_i is isomorphic to H as a permutation group.

Suppose that $\Omega = C_1 \cup \cdots \cup C_\ell$ is another nontrivial system of imprimitivity for G. We will show that $\{B_1, \ldots, B_k\}$ is a refinement of $\{C_1, \ldots, C_\ell\}$, which proves the lemma. Each element of C_1 is contained in some B_i, so without loss of generality we can assume that $C_1 \cap B_1 \neq \emptyset$.

Consider first the case where $C_1 \not\subseteq B_1$. Then there exists $y \in C_1 \cap B_j$ for some $j \neq 1$. For $g \in H_1$ we have $g(y) = y$, so in particular $g(C_1) = C_1$. Thus H_1 acts on C_1. Because H_1 is transitive on B_1 and $C_1 \cap B_1 \neq \emptyset$, it follows that $B_1 \subseteq C_1$. By the same argument, we have $B_i \subseteq C_1$ for any i such that $C_1 \cap B_i \neq \emptyset$. Thus $C_1 = B_{i_1} \cup \cdots \cup B_{i_t}$ for some indices $i_1 < \cdots < i_t$, so $\{B_1, \ldots, B_k\}$ is a refinement of $\{C_1, \ldots, C_\ell\}$.

Suppose then that $C_1 \subseteq B_1$. Since H_1 acts on $\{C_1, \ldots, C_\ell\}$ and since H_1 is transitive on B_1, it follows that $B_1 = C_{j_1} \cup \cdots \cup C_{j_s}$ for some indices $j_1 < \cdots < j_s$. Because $\{C_1, \ldots, C_\ell\}$ is a nontrivial system of imprimitivity, by primitivity of H_1 we must have $s = 1$ and $B_1 = C_1$. In this case $\{B_1, \ldots, B_k\} = \{C_1, \ldots, C_\ell\}$, as required. □

Recall the following result, which goes back to the work of Galois [**67**] — see for example [**38**, II, Satz 3.2] and [**18**, Theorem 4.7A].

Proposition 1.3.4. *Let $G \leq \mathrm{Sym}(\Omega)$ be primitive and solvable, where $|\Omega| = N > 1$. Then the following statements hold:*

(i) *$N = p^n$ for some prime p and integer $n > 0$;*
(ii) *G has a unique minimal normal subgroup K;*
(iii) *K is transitive and regular, and $K \cong C_p^n$;*
(iv) *G is a semidirect product $G = K \rtimes G_\omega$ for all $\omega \in \Omega$;*
(v) *All complements to K in G are conjugate in G;*
(vi) *As a permutation group G is isomorphic to an affine group $V \rtimes X$, where $V = \mathbb{F}_p^n$, and $X \leq \mathrm{GL}_n(p)$ is irreducible and solvable.*

For the proof of Theorem 1.1.2, we will need the following observation from [**52**, p. 272].

Lemma 1.3.5. *Let G be a primitive solvable subgroup of* $\mathrm{Sym}(\Omega)$. *Then for all $g \in G \setminus \{1\}$, we have $|\Omega^g| \leq |\Omega|/2$.*

Proof. Since G is primitive solvable, we have $|\Omega| = p^n$ for some prime p and integer $n > 0$ (Proposition 1.3.4). Moreover, as a permutation group G is isomorphic to $V \rtimes X \leq \mathrm{Sym}(V)$, where V is an n-dimensional vector space over $\mathbb{F}_p$, and $X \leq \mathrm{GL}(V)$ is irreducible. We can identify $V \rtimes X$ as the set of affine linear transformations

$$\{\varphi_{A,w} : V \to V : A \in X \text{ and } w \in V\} \leq \mathrm{Sym}(V)$$

where $\varphi_{A,w}(v) = Av + w$ for all $v \in V$.

The set of fixed points for $\varphi_{A,w}$ is either empty, or equal to the affine subspace $v_0 + V^A$ for some $v_0 \in V$. We have $|v_0 + V^A| = p^{n'}$ where $n' = \dim(V^A)$, so

$$|\Omega^g| \leq |V^A| \leq p^{n-1} \leq |\Omega|/2$$

if $A \neq 1$. If $A = 1$, then $\varphi_{A,w}$ has no fixed points on V unless $w = 0$, in which case $\varphi_{A,w}$ is the identity. □

Proof of Theorem 1.1.1. Let $\Omega = \{1, 2, \ldots, N\}$. Suppose first that G is intransitive. Let

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_t$$

be the decomposition of Ω into G-orbits, so $t \geq 2$. Set $n_i = |\Omega_i|$, so $N = n_1 + n_2 + \cdots + n_t$, where $n_i \geq 1$ for all $1 \leq i \leq t$. Then $G \leq G_1 \times G_2 \times \cdots \times G_t$ where $G_i \leq S_{n_i}$ is the action of G on Ω_i, so $G = G_1 \times G_2 \times \cdots \times G_t$ since G is maximal solvable. Moreover, each G_i must be maximal transitive solvable since G is maximal solvable.

We next check that conditions (c) – (e) hold for G. To this end, suppose that there exists $i \neq j$ such that $G_i \cong G_j \wr S_k$ for some $1 \leq k \leq 3$. Then G is not maximal solvable, since $(G_j \wr S_k) \times G_j \lneqq G_j \wr S_{k+1}$. Therefore (c) – (e) must hold.

Next we consider the case where G is transitive and imprimitive. Let $\Omega = B_1 \cup B_2 \cup \cdots \cup B_r$ be a system of imprimitivity for G, where $1 < r < N$. Then $G \leq N_G(B_1) \wr X$, where $N_G(B_1) \leq S_{N/r}$ is the stabilizer of the block B_1 in G, and $X \leq S_r$ is the action of G on $\{B_1, B_2, \ldots, B_r\}$. Since G is transitive and maximal solvable, we have $G = N_G(B_1) \wr X$ and both $N_G(B_1)$ and X are maximal transitive solvable subgroups of $S_{N/r}$ and S_r, respectively.

Repeating this argument with $N_G(B_1)$ and X and applying induction, it follows that $G = G_1 \wr G_2 \wr \cdots \wr G_t$, where $G_i \leq S_{n_i}$ is maximal primitive solvable ($n_i > 1$), and $N = n_1 n_2 \cdots n_t$. Solvable primitive groups only occur in prime power degree, so each n_i is a prime power. Moreover if $(n_i, n_{i+1}) = (2, 2)$ for some $1 \leq i < t$, then G is not maximal solvable, since $S_2 \wr S_2 \lneq S_4$.

It remains to consider the case where G is transitive and primitive. This case follows from Proposition 1.3.4. $\square$

REMARK 1.3.6. In Theorem 1.1.1, for groups of Type (I) it is possible that $n_i = 1$, but by (I.c) this holds for at most one i.

We also note that (I.d) excludes $G = S_2 \times S_1 = (S_1 \wr S_2) \times S_1$, which is not maximal solvable in S_3. Similarly (I.e) excludes $G = S_3 \times S_1 = (S_1 \wr S_3) \times S_1$, which is not maximal solvable in S_4. However, note that for example the point stabilizer $S_4 \times S_1$ is maximal in S_5, and thus in particular maximal solvable.

PROOF OF THEOREM 1.1.2. By induction on $N = |\Omega|$. There is nothing to prove for $N = 1$, so assume that $N > 1$. If G is not maximal solvable, then by Theorem 1.1.1 we have $G \lneq \overline{G} \leq S_N$, where $\overline{G}$ is of one of the types (I) – (III). We consider the possibilities for the types of G and $\overline{G}$ in turn, and will mostly argue similarly to [**52**, §9 – §14].

Case 1: G is of type (I).
In this case Ω decomposes as a disjoint union of G-orbits

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_t,$$

where $t \geq 2$. Moreover $G = G_1 \times G_2 \times \cdots \times G_t$, where $G_i \leq \mathrm{Sym}(\Omega_i)$ is maximal transitive solvable.

Case 1.1: $\overline{G}$ is of type (I).
We have $\overline{G} = \overline{G_1} \times \overline{G_2} \times \cdots \times \overline{G_s}$ and $\Omega = \overline{\Omega_1} \cup \overline{\Omega_2} \cup \cdots \cup \overline{\Omega_s}$ with $\overline{G_i} \leq \mathrm{Sym}(\overline{\Omega_i})$ maximal transitive solvable. Each $\overline{\Omega_i}$ is a union of some G-orbits, so there exist indices $1 \leq j_1 < \cdots < j_k \leq t$ such that $G_{j_1} \times G_{j_2} \times \cdots \times G_{j_k} \leq \overline{G_i}$. If $k > 1$, then by induction $G_{j_1} \times G_{j_2} \times \cdots \times G_{j_k}$ is maximal solvable, so $G_{j_1} \times G_{j_2} \times \cdots \times G_{j_k} = \overline{G_i}$, contradicting the transitivity of $\overline{G_i}$. Thus $k = 1$, and so we have $G_{j_1} = \overline{G_i}$, since each G_j is maximal solvable. Since this holds for all i, we conclude $G = \overline{G}$, contrary to our assumption $G \lneq \overline{G}$.

Case 1.2: $\overline{G}$ is of type (II).
Here $\overline{G}$ can be written as a wreath product $\overline{G} = X \wr \Delta$, where $\Delta \leq S_e$ is primitive solvable, $X \leq S_d$ is transitive solvable, and $N = de$. Let $\Omega = \overline{B_1} \cup \cdots \cup \overline{B_e}$ be the corresponding system of imprimitivity, so now $\overline{G} = (X_1 \times \cdots \times X_e) \rtimes \Delta$, where the X_i are isomorphic to X as permutation groups.

Without loss of generality we may assume that $\Omega_1 \cap \overline{B_1} \neq \emptyset$. Suppose first that $\overline{B_1} \not\subseteq \Omega_1$. Then there exists some $i \neq 1$ such that $\Omega_i \cap \overline{B_1} \neq \emptyset$. Arguing as in the proof of Lemma 1.3.3 (paragraph 3) shows that $\Omega_i \subseteq \overline{B_1}$ for any i such that $\Omega_i \cap \overline{B_1} \neq \emptyset$. Thus $\overline{B_1} = \Omega_{i_1} \cup \cdots \cup \Omega_{i_k}$ for some indices $i_1 < \cdots < i_k$, so $G_{i_1} \times \cdots \times G_{i_k}$ is contained in X_1, with $k \geq 2$. But this is a contradiction, since $G_{i_1} \times \cdots \times G_{i_k}$ is maximal solvable by induction, and X_1 is transitive solvable.

It follows then that $\overline{B_1} \subseteq \Omega_1$. The group G acts on $\{\overline{B_1}, \ldots, \overline{B_e}\}$ since $\overline{G}$ does, so $\Omega_1 = \overline{B_{j_1}} \cup \cdots \cup \overline{B_{j_{\ell_1}}}$ for some $j_1 < \cdots < j_{\ell_1}$ and $\ell_1 \geq 1$. Since G_1 is maximal solvable, it follows that $G_1 = X \wr \Delta_1$ for some maximal transitive solvable

$\Delta_1 \leq S_{\ell_1}$. For $i > 1$ the same arguments show that $\overline{B_i}$ is contained in some Ω_j, and $G_i = X \wr \Delta_i$ for some maximal transitive solvable $\Delta_i \leq S_{\ell_i}$, where $\ell_i \geq 1$.

Thus we have $G = X \wr \Delta_1 \times \cdots \times X \wr \Delta_t$. Now $\Delta_1 \times \cdots \times \Delta_t$ is a type (I) subgroup of S_e because G is, so by induction it is maximal solvable. But this is a contradiction, since we have $\Delta_1 \times \cdots \times \Delta_t \leq \Delta$, and Δ is transitive solvable.

Case 1.3: $\overline{G}$ is of type (III).
Now $N = p^n$ for some prime p and integer $n > 0$, and $\overline{G}$ is primitive. By Proposition 1.3.4, the group $\overline{G}$ has a unique minimal normal subgroup $\overline{K}$, with $\overline{K} \cong (C_p)^n$. Moreover for every $\omega \in \Omega$, we have $\overline{G} = \overline{K} \rtimes \overline{G}_\omega$, where $\overline{G}_\omega \leq \mathrm{GL}_n(p)$.

By Lemma 1.3.5, each $g \in \overline{G} \setminus \{1\}$ has at most $N/2$ fixed points. We first use this fact to show that we must have $t = 2$. To this end, note first that by property (I.c) at most one G_i is trivial. Thus if $t > 2$, there exists some $1 \leq i \leq t$ such that $1 < |\Omega_i| < N/2$. But since G_i fixes every point in $\Omega \setminus \Omega_i$, a nontrivial element of G_i has $> N/2$ fixed points, which is a contradiction. Thus $t = 2$, and $G = G_1 \times G_2$.

If G_i is nontrivial, the previous argument shows that $|\Omega_i| \geq N/2$. Thus if G_1 and G_2 are both nontrivial, we have $|\Omega_1| = |\Omega_2| = N/2$, and every nontrivial element of G_i has no fixed points on Ω_i. Furthermore in this case N is even, so $p = 2$. Nontrivial groups of type (I) and (II) contain nontrivial elements with fixed points, so G_1 and G_2 are both primitive of type (III). In other words $G_i = (C_2)^{n-1} \rtimes X_i$ with $X_i \leq \mathrm{GL}_{n-1}(2)$ maximal irreducible solvable. But the point stabilizer X_i of G_i is trivial, so $\mathrm{GL}_{n-1}(2)$ is trivial. Therefore $p = n = 2$ and $G_1 \cong G_2 \cong S_2$ as permutation groups, contrary to (I.c).

It remains to consider the case where either G_1 or G_2 is trivial. Without loss of generality we assume that $G_2 = \{1\}$. Then G_1 cannot be maximal solvable of type (I) or (II). Indeed, otherwise G_1 would contain a nontrivial element with $\geq |\Omega_1|/2 = (N-1)/2$ fixed points on Ω_1, and thus G_1 would contain a nontrivial element with $> N/2$ fixed points on Ω.

Therefore G_1 is a primitive solvable subgroup of type (III) in $\mathrm{Sym}(\Omega_1)$, so $p^n - 1 = r^\ell$ for some prime r and integer $\ell > 0$. Since G_1 fixes the point in $\Omega_2 = \{\omega\}$, we have $G_1 \leq \overline{G}_\omega \leq \mathrm{GL}_n(p)$. Now G_1 contains a minimal normal subgroup $K \cong (C_r)^\ell$ which is transitive and regular on Ω_1 (Proposition 1.3.4).

Thus as a subgroup of $\mathrm{GL}_n(p)$, the minimal normal subgroup K acts transitively on the nonzero vectors in $\mathbb{F}_p^n$. In particular K is irreducible. Since K is abelian, it follows from Schur's lemma that K is a subgroup of the multiplicative group of a finite field, and in particular K is cyclic. Therefore $\ell = 1$, and in this case $G_1 \leq S_r$ is the normalizer of a r-cycle and $|G_1| = r(r-1)$.

On the other hand K is an irreducible cyclic subgroup of $\mathrm{GL}_n(p)$ with order $r = p^n - 1$, so it is generated by a Singer cycle; see [**38**, II, Satz 3.10] and [**38**, II, Satz 7.3]. Thus as a subgroup of $\mathrm{GL}_n(p)$, the group G_1 lies in the normalizer of a Singer cycle. The normalizer of a Singer cycle in $\mathrm{GL}_n(p)$ has order $n(p^n-1)$ by [**38**, II, Satz 7.3 (a)], so it follows that $|G_1| = (p^n-1)(p^n-2) \leq n(p^n-1)$ and thus

$$p^n - 2 \leq n.$$

By this inequality, one of the following holds:

- $n = 1$ and $p = 3$, $r = 2$;
- $n = 2$ and $p = 2$, $r = 3$.

In the first case $G = S_2 \times S_1$ and in the second case $G = S_3 \times S_1$. But these cases are excluded by (I.d) and (I.e), so we have a contradiction.

Case 2: G is of type (II).
By induction, in this case G can be written as a wreath product $G = \Gamma \wr \Delta$, where $\Delta \leq S_t$ is maximal transitive solvable, $\Gamma \leq S_d$ is maximal primitive solvable, and $N = dt$. Let $\Omega = B_1 \cup \cdots \cup B_t$ be the corresponding system of imprimitivity. Then $G = (\Gamma_1 \times \cdots \times \Gamma_t) \rtimes \Delta$, where $\Gamma_i \leq \mathrm{Sym}(B_i)$ is isomorphic to Γ as a permutation group for all $1 \leq i \leq t$.

Case 2.1: $\overline{G}$ is of type (II).
In this case we can write $\overline{G} = \overline{\Gamma} \wr \overline{\Delta}$ for some $\overline{\Gamma}, \overline{\Delta} \neq \{1\}$ maximal transitive solvable, such that $\overline{\Gamma}$ is primitive. Let $\Omega = \overline{B_1} \cup \cdots \cup \overline{B_s}$ be the corresponding system of imprimitivity for $\overline{G}$, so $\overline{G} = (\overline{\Gamma_1} \times \cdots \times \overline{\Gamma_s}) \rtimes \overline{\Delta}$, where each $\overline{\Gamma_i} \leq \mathrm{Sym}(\overline{B_i})$ is isomorphic as a permutation group to $\overline{\Gamma}$.

Without loss of generality, we can assume that $B_1 \cap \overline{B_1} \neq \emptyset$. It follows from Lemma 1.3.3 that $\{B_1, \ldots, B_t\}$ is a refinement of $\{\overline{B_1}, \ldots, \overline{B_s}\}$. Thus $\overline{B_1} = B_{i_1} \cup \cdots \cup B_{i_k}$ for some $1 = i_1 < \cdots < i_k$, so $\Gamma_{i_1} \times \cdots \times \Gamma_{i_k} \leq \overline{\Gamma_1}$.

Suppose first that $k > 1$. Since $\overline{\Gamma_1}$ is primitive and $\Gamma \neq 1$, by applying Lemma 1.3.5 as in Case 1.3, it follows that $k = 2$ and $\Gamma \cong S_2$. Then $\overline{B_1} = B_{i_1} \cup B_{i_2}$, so $\Delta \leq S_2 \wr \Delta'$, where Δ' is the action of G on $\{\overline{B_1}, \ldots, \overline{B_s}\}$. Because Δ is maximal solvable, it follows that $\Delta = S_2 \wr \Delta'$. But then $G = S_2 \wr S_2 \wr \Delta'$, which is excluded by (II.c).

Therefore $k = 1$, in which case $\overline{B_1} = B_1$. Then $\Gamma_1 \leq \overline{\Gamma_1}$, so by maximality of Γ_1 we have $\Gamma_1 = \overline{\Gamma_1}$. After rearranging the factors if necessary, it follows that $\Gamma_i = \overline{\Gamma_i}$ for all $1 \leq i \leq t$. Then $G = \Gamma \wr \Delta$ and $\overline{G} = \Gamma \wr \overline{\Delta}$, where $\Delta \leq \overline{\Delta}$. Since Δ is maximal solvable we have $\Delta = \overline{\Delta}$, so in fact $G = \overline{G}$, contrary to our assumption $G \lneqq \overline{G}$.

Case 2.2: $\overline{G}$ is of type (III).
As in Case 1.3, we must have $t = 2$, as otherwise nontrivial elements of Γ_i would have $> N/2$ fixed points on Ω, contradicting Lemma 1.3.5. Thus $G = \Gamma \wr S_2 = (\Gamma_1 \times \Gamma_2) \rtimes S_2$. Since Γ_1 fixes the $N/2$ points in B_2, by Lemma 1.3.5 nontrivial elements of Γ_1 have no fixed points on B_1. As in Case 1.3, it follows that Γ_1 is primitive and $\Gamma_1 \cong S_2$. But then $G = S_2 \wr S_2$, which is excluded by (II.c).

Case 3: G is of type (III).
In this case $N = p^n$ for some prime p and integer $n > 0$. By Proposition 1.3.4, the group G has a unique minimal normal subgroup K, which is a transitive regular elementary abelian subgroup isomorphic to $(C_p)^n$. Furthermore G is a semidirect product $G = KG_\omega$ for every $\omega \in \Omega$, and G_ω is identified as a maximal irreducible solvable subgroup of $\mathrm{Aut}(K) = \mathrm{GL}_n(p)$.

Since G is primitive, the same must be true for $\overline{G}$. Thus similarly by Proposition 1.3.4, the group $\overline{G}$ has a unique minimal normal subgroup $\overline{K}$, and $\overline{G} = \overline{K}\ \overline{G}_\omega$ with $\overline{G}_\omega \leq \mathrm{GL}_n(p)$.

We will show that $K = \overline{K}$. To this end, consider first the possibility that $K \cap \overline{K} = \{1\}$. If this is the case, then we can consider K as a subgroup of $\mathrm{GL}_n(p)$. The number of nonzero vectors in $V = (\mathbb{F}_p)^n$ is $p^n - 1$ which is coprime to p, so K fixes some nonzero vector in V. Now G acts on V^K since K is a normal subgroup.

On the other hand G is primitive, so it must act irreducibly on V. Therefore $V^K = V$, which is a contradiction since $K \neq \{1\}$.

Therefore we must have $K \cap \overline{K} \neq \{1\}$, and so $K \leq \overline{K}$ since K is minimal normal. But $|K| = |\overline{K}| = p^n$, so $K = \overline{K}$. In this case $G = KG_\omega$ and $\overline{G} = K\overline{G}_\omega$ with

$$G_\omega \leq \overline{G}_\omega \leq \mathrm{GL}_n(p).$$

Since G_ω is assumed to be maximal solvable in $\mathrm{GL}_n(p)$, we have $G_\omega = \overline{G}_\omega$. Then $G = \overline{G}$, which is a contradiction. □

We next describe conjugacy among the subgroups of S_N described in Theorem 1.1.1. Since transitivity and primitivity is preserved by conjugacy, it suffices to do this for groups of the same type. Results similar to Propositions 1.3.8 and 1.3.9 below were also observed in [**74**, Theorem 2.1.4, Theorem 2.1.6].

Proposition 1.3.7. *Suppose that $G = G_1 \times \cdots \times G_t$ and $\overline{G} = \overline{G_1} \times \cdots \times \overline{G_s}$ are of type (I) as in Theorem 1.1.1. Then G and $\overline{G}$ are conjugate in S_N if and only if all of the following hold:*

(i) *$t = s$;*
(ii) *There exists a permutation π of $\{1, \ldots, t\}$ such that $G_i \cong \overline{G_{\pi(i)}}$ as permutation groups for all $1 \leq i \leq t$.*

Proof. Sufficiency is clear. For the other direction, suppose that $gGg^{-1} = \overline{G}$ for $g \in S_N$. Let $\{\Omega_1, \ldots, \Omega_t\}$ and $\{\overline{\Omega_1}, \ldots, \overline{\Omega_s}\}$ be the orbits of G and $\overline{G}$, respectively. Then $\{g\Omega_1, \ldots, g\Omega_t\}$ are the orbits of $\overline{G}$, so $t = s$ and there exists a permutation π of $\{1, \ldots, t\}$ such that $g\Omega_i = \overline{\Omega_{\pi(i)}}$ for all $1 \leq i \leq t$. It follows that $G_i \cong \overline{G_{\pi(i)}}$ as permutation groups for all $1 \leq i \leq t$. □

Proposition 1.3.8. *Suppose that $G = G_1 \wr \cdots \wr G_t$ and $\overline{G} = \overline{G_1} \wr \cdots \wr \overline{G_s}$ are of type (II) as in Theorem 1.1.1. Then G and $\overline{G}$ are conjugate in S_N if and only if all of the following hold:*

(i) *$t = s$;*
(ii) *$G_i \cong \overline{G_i}$ as permutation groups for all $1 \leq i \leq t$.*

Proof. Sufficiency is straightforward. For the other direction, suppose that $gGg^{-1} = \overline{G}$ for some $g \in S_N$. Write $G = H \wr K$ with $H = G_1$ and $K = G_2 \wr \cdots \wr G_t$, and similarly $\overline{G} = \overline{H} \wr \overline{K}$ with $\overline{H} = \overline{G_1}$ and $\overline{K} = \overline{G_2} \wr \cdots \wr \overline{G_s}$.

Let $\{B_1, \ldots, B_k\}$ be the system of imprimitivity defining $G = H \wr K$, which is nonrefinable since H is primitive. Then $\{gB_1, \ldots, gB_k\}$ is a nonrefinable system of imprimitivity for $\overline{G}$, so by Lemma 1.3.3 it is the system of imprimitivity defining $\overline{G} = \overline{H} \wr \overline{K}$.

The action of $N_G(B_1)$ on B_1 is isomorphic to H as a permutation group, while the action of $N_{\overline{G}}(gB_1)$ on gB_1 is isomorphic to $\overline{H}$ as a permutation group. On the other hand $gN_G(B_1)g^{-1} = N_{\overline{G}}(gB_1)$, so $H \cong \overline{H}$ as permutation groups.

The action of K on $\{B_1, \ldots, B_k\}$ is faithful and isomorphic as a permutation group to the action of gKg^{-1} on $\{gB_1, \ldots, gB_k\}$. It follows that $K \cong gKg^{-1} \cong \overline{K}$ as permutation groups. If $t = 2$ we are done, and for $t > 2$ the result follows by induction on t. □

Proposition 1.3.9. *Suppose that $N = p^n$ and that $G = (C_p)^n \rtimes X$ and $\overline{G} = (C_p)^n \rtimes \overline{X}$ are of type (III) as in Theorem 1.1.1. Then G and $\overline{G}$ are conjugate in S_N if and only if X and $\overline{X}$ are conjugate in $\mathrm{GL}_n(p)$.*

PROOF. Follows from Proposition 1.3.4 (ii) and (v). □

With Theorem 1.1.1 – 1.1.2 and Proposition 1.3.7 – 1.3.9, the problem of classifying maximal solvable subgroups of S_N is completely reduced to the problem of classifying maximal irreducible solvable subgroups of $\mathrm{GL}_n(p)$.

EXAMPLE 1.3.10. We illustrate Theorems 1.1.1 and 1.1.2 in Table 1.1, where we list all maximal solvable subgroups of S_n up to conjugacy, for $5 \leq n \leq 10$. In the table, we use the notation

$$\mathrm{AGL}_k(p) := \mathbb{F}_p^k \rtimes \mathrm{GL}_k(p)$$

for the affine permutation group corresponding to $\mathrm{GL}_k(p)$.

The group $\Gamma\mathrm{L}_1(2^3)$ that appears in case $n = 8$ is the normalizer of a Singer cycle in $\mathrm{GL}_3(2)$. It is not too difficult to see that this is the unique maximal irreducible solvable subgroup of $\mathrm{GL}_3(2)$ up to conjugacy; this fact will also follow from results proven in later sections. For maximal transitive solvable subgroups, we give more examples in Section 8.1, Table 8.1.

For $1 \leq n \leq 4$ the symmetric group S_n is solvable, so in these small cases the only maximal solvable subgroup of S_n is the group itself, which is primitive. In terms of the classification in Theorems 1.1.1 and 1.1.2, we have

$$\begin{aligned} S_2 &= \mathbb{F}_2 \rtimes \mathrm{GL}_1(2) = \mathrm{AGL}_1(2), \\ S_3 &= \mathbb{F}_3 \rtimes \mathrm{GL}_1(3) = \mathrm{AGL}_1(3), \\ S_4 &= \mathbb{F}_2^2 \rtimes \mathrm{GL}_2(2) = \mathrm{AGL}_2(2), \end{aligned}$$

as permutation groups.

EXAMPLE 1.3.11. Suppose that $n > 1$ is squarefree, so $n = p_1 \dots p_t$ with p_1, …, p_t distinct primes. For each $1 \leq i \leq t$, the permutation group

$$X_i := \mathbb{F}_{p_i} \rtimes \mathrm{GL}_1(p_i) = \mathrm{AGL}_1(p_i)$$

is the unique maximal primitive solvable subgroup of S_{p_i}. By Theorems 1.1.1 and 1.1.2, the maximal transitive solvable subgroups of S_n are precisely the subgroups of the form

$$X_{p_{\pi(1)}} \wr \cdots \wr X_{p_{\pi(t)}},$$

where π is some permutation of $\{1, \dots, t\}$. In particular, in this case S_n has exactly $t!$ maximal transitive solvable subgroups, up to conjugacy (Proposition 1.3.8 – 1.3.9). This example was also observed by Jordan in [**48**, Table B, p. 288] and by Suprunenko in [**76**, Example 4(b), pp. 49–50].

EXAMPLE 1.3.12. Suppose that $n > 1$ is of the form $n = 4p_1 \cdots p_t$, with p_1, …, p_t distinct odd primes. Then it follows from Theorem 1.1.1, Theorem 1.1.2, and Proposition 1.3.8 – 1.3.9 that S_n has exactly $(t+2)!/2$ maximal transitive solvable subgroups, up to conjugacy. (This is also stated by Jordan in [**48**, Table B, p. 288].)

For example in the case where $t = 2$ with $n = 4p_1p_2$, representatives for the 12 conjugacy classes of maximal transitive solvable subgroups are given by

$$\begin{array}{ccc} S_4 \wr X \wr Y & X \wr S_4 \wr Y & X \wr Y \wr S_4 \\ S_2 \wr X \wr S_2 \wr Y & S_2 \wr X \wr Y \wr S_2 & X \wr S_2 \wr Y \wr S_2 \end{array}$$

where $\{X, Y\} = \{\mathrm{AGL}_1(p_1), \mathrm{AGL}_1(p_2)\}$.

TABLE 1.1. Maximal solvable subgroups of S_n for $5 \leq n \leq 10$, up to conjugacy in S_n. (See Example 1.3.10.)

n	X	$\lvert X \rvert$	type
5	$\mathrm{AGL}_1(5)$	20	primitive
	$S_4 \times S_1$	24	intransitive
	$S_3 \times S_2$	12	intransitive
6	$S_3 \wr S_2$	72	imprimitive
	$S_2 \wr S_3$	48	imprimitive
	$\mathrm{AGL}_1(5) \times S_1$	20	intransitive
	$S_4 \times S_2$	48	intransitive
7	$\mathrm{AGL}_1(7)$	42	primitive
	$(S_3 \wr S_2) \times S_1$	72	intransitive
	$(S_2 \wr S_3) \times S_1$	48	intransitive
	$\mathrm{AGL}_1(5) \times S_2$	40	intransitive
	$S_4 \times S_3$	144	intransitive
8	$\mathbb{F}_2^3 \rtimes \Gamma\mathrm{L}_1(2^3)$	168	primitive
	$S_4 \wr S_2$	1152	imprimitive
	$S_2 \wr S_4$	384	imprimitive
	$(S_3 \wr S_2) \times S_2$	144	intransitive
	$\mathrm{AGL}_1(5) \times S_3$	120	intransitive
	$\mathrm{AGL}_1(7) \times S_1$	42	intransitive
9	$\mathrm{AGL}_2(3)$	432	primitive
	$S_3 \wr S_3$	1296	imprimitive
	$(\mathbb{F}_2^3 \rtimes \Gamma\mathrm{L}_1(2^3)) \times S_1$	168	intransitive
	$(S_4 \wr S_2) \times S_1$	1152	intransitive
	$(S_2 \wr S_4) \times S_1$	384	intransitive
	$\mathrm{AGL}_1(7) \times S_2$	84	intransitive
	$(S_2 \wr S_3) \times S_3$	288	intransitive
	$\mathrm{AGL}_1(5) \times S_4$	480	intransitive
	$S_4 \times S_3 \times S_2$	288	intransitive
10	$\mathrm{AGL}_1(5) \wr S_2$	800	imprimitive
	$S_2 \wr \mathrm{AGL}_1(5)$	640	imprimitive
	$\mathrm{AGL}_2(3) \times S_1$	432	intransitive
	$(S_3 \wr S_3) \times S_1$	1296	intransitive
	$\mathrm{AGL}_1(5) \times S_4 \times S_1$	480	intransitive
	$(\mathbb{F}_2^3 \rtimes \Gamma\mathrm{L}_1(2^3)) \times S_2$	336	intransitive
	$(S_4 \wr S_2) \times S_2$	2304	intransitive
	$(S_2 \wr S_4) \times S_2$	768	intransitive
	$\mathrm{AGL}_1(5) \times S_3 \times S_2$	240	intransitive
	$\mathrm{AGL}_1(7) \times S_3$	252	intransitive
	$(S_3 \wr S_2) \times S_4$	1728	intransitive
	$(S_2 \wr S_3) \times S_4$	1152	intransitive

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